JOURNAL OF APPROXIMATION THEORY 46, 51–57 (1986)

Géza Freud and Lacunary Fourier Series

JEAN-PIERRE KAHANE

Université de Paris-Sud, Equipe de Recherche Associée au CNRS (757), Analyse Harmonique, Mathématique (Bât. 425), 91405 Orsay Cedex, France

Communicated by Paul G. Nevai

Received May 4, 1984

DEDICATED TO THE MEMORY OF GÉZA FREUD

There is only one paper of Géza Freud devoted to lacunary Fourier series [2], and part of another [1]. They are beautiful, and I am happy to have an opportunity to write on them.

Let us consider an Hadamard lacunary Fourier series

$$f(x) \sim \sum_{k=0}^{\infty} (a_k \cos n_k x + b_k \sin n_k x), \tag{1}$$

where $n_0 = 0$, $n_1 \geqslant 1$, and $\inf_{k \geqslant 1}(n_{k+1}/n_k) = q > 1$. It may be convenient to write the series in (1) in the form

$$\sum_{k=0}^{\infty} r_k \cos(n_k x + \varphi_k) \qquad (r_k \geqslant 0) \tag{2}$$

or

$$\sum_{-\infty}^{\infty} c_k e^{in_k x} \qquad (n_{-k} = -n_k). \tag{3}$$

If $f(x)$ is bounded on an interval, the three series converge absolutely, and we can write = instead of ~ in (1). The first example of such series is the famous Weierstrass example of a nowhere differentiable function ($n_k = b^k$, where b is an odd integer $\geqslant 3$, and $r_k = a^k = a_k$ with $0 < a < 1$ and $ab > 1 + (3\pi/2)$). Weierstrass's condition on ab is just for convenience, and it can be relaxed to $ab \geqslant 1$. This was discovered by Hardy in 1916, using a very complicated method. Actually Géza Freud made it quite simple (Theorem 6 in his 1962 paper).

THEOREM 1. *If $f(x)$ is differentiable at one point, $r_k = o(n_k^{-1})$ $(k \to \infty)$.*

0021-9045/86 \$3.00

This was the starting point both of his 1966 paper, and of a work of M. and S. Izumi and myself [3]. Let me explain the I. I. K. point of view, which goes back to M. E. Noble [4] and Masako Sato (Mrs. Izumi) [5]. The Fourier coefficients in (3) can be written as

$$c_k = \int_{-\pi}^{\pi} f(x)\, T_k(x)\, e^{-in_k x}\, dx \Big/ \int_{-\pi}^{\pi} T_k(x)\, dx \tag{4}$$

whenever $T_k(x)$ is a positive trigonometric polynomial of degree smaller than ν_k, given by

$$\nu_k = \min(n_{k+1} - n_k, n_k - n_{k-1}).$$

The same is true if $k \geqslant 2$ and if we replace $f(x)$ by $f(x) - a\cos(x+\varphi)$. If $f(x)$ is differentiable at 0, we can choose a and φ so that

$$f(x) - a\cos(x+\varphi) = o(|x|) \qquad (x \to 0),$$

therefore, assuming that $\int_{-\delta}^{\delta} T_k / \int_{-\pi}^{\pi} T_k$ converges to 1 $(k \to \infty)$ whatever $\delta > 0$ is,

$$c_k = o\left(\int_{-\pi}^{\pi} |x|\, T_k(x)\, dx \Big/ \int_{-\pi}^{\pi} T_k(x)\, dx\right).$$

Choosing $T_k =$ square of the Fejér kernel of degree $[\nu_k/2]$ we obtain $c_k = o(\nu_k^{-1})$, therefore $c_k = o(n_k^{-1})$. The same conclusion holds if we assume that $f(x)$ is differentiable at any given point, which proves Freud's theorem.

This works in an even simpler way if we assume that $f(x)$ satisfies a Lipschitz–Hölder condition at a point x_0, i.e.,

$$f(x_0 + h) - f(x_0) = O(|h|^{\alpha}) \qquad (h \to 0, 0 < a < 1).$$

Then the conclusion is

$$r_k = O(n_k^{-\alpha}),$$

which, in turn, implies $f \in \operatorname{Lip} \alpha$. Therefore, for Hadamard lacunary Fourier series, a Lipschitz condition of order α at one point (note that $0 < \alpha < 1$!) implies the same uniform Lipschitz condition. Is it possible to replace the Hadamard lacunary condition by another? No, it is one of the few examples where the Hadamard lacunary condition proves necessary as well as sufficient. This is in the I.I.K. paper (see also Kahane [26]), together with a number of variations on the same theme (consider a C^k or even a C^∞ function at *one* point, estimate the successive derivatives, find the smallest function near 0 with spectrum $\{n_k\}$, etc.). This direction was continued by Benke [6].

While I.I.K. start from (4) Geza Freud systematically uses the relations between

$$\frac{f(x+h)-f(x)}{h}, \qquad s_n'(x), \sigma_n'(x),$$

where s_n and σ_n are the partial and Fejér sums of the Fourier series, and $h \approx 1/n$. This idea was elaborated in his 1962 paper [1], without assuming first any kind of lacunarity. Here is one of the results he obtains in 1966: suppose $\psi(\delta)$ is an increasing function $(\delta > 0)$ such that

$$\int_0^\delta \frac{\psi(t)}{t}\,dt + \delta \int_\delta^\pi \frac{\psi(t)}{t^2}\,dt = O(1)\,\psi(\delta).$$

If $f(x)$ is the function in (1) and satisfies

$$f(x_0+h)-f(x_0) = O(1)\,\psi(|h|) \qquad (\text{resp. } o(1)\,\psi(|h|))$$

as $|h| \to 0$ at one single point x_0, then we have uniformly in x

$$f(x+h)-f(x) = O(1)\,\psi(|h|) \qquad (\text{resp. } o(1)\,\psi(|h|)).$$

When $\psi(t) = t^\alpha$ $(0 < \alpha < 1)$, it is what we already mentioned. En passant, he obtains estimates for $\Sigma_{k \geqslant N} r_k$ or $\sum_{k \geqslant N} n_k r_k$ under various local conditions for $f(x)$ at x_0.

Let us go back to the 1962 theorem. What happens if $r_k = 1/n_k$? We already know that $f(x)$ is nowhere differentiable. Can we say more? Yes, indeed, and it is a beautiful statement of Geza Freud (Theorem 5), of which I am giving now a slight improvement.

THEOREM 2. *Suppose* $0 < \alpha \leqslant n_k r_k \leqslant \beta < \infty$ *and* $n_k \leqslant \nu^k$ $(k = 1, 2, \ldots)$. *Then*

(a) $f(x+h)+f(x-h)-2f(x) = O(|h|)$ *and* $f(x+h)-f(x) = O(|h| \log 1/|h|)$ $(h \to 0)$ *uniformly with respect to* x $(x \in [-\pi, \pi])$.

(b) $\overline{\lim}_{h \to 0} |f(x+h)-f(x)|/(|h| \log 1/|h|) > 0$ *for every* x *except on a set of the first category of Baire.*

(c) $0 < \overline{\lim}_{h \to 0} |f(x+h)-f(x)|/(|h| \sqrt{\log 1/h \log\log\log 1/h}) < \infty$ *for almost every* x.

(d) $\underline{\lim}_{h \to 0} |f(x+h)-f(x)|/|h| < \infty$ *for a dense set of* x.

(e) $f(x)$ *is nowhere differentiable.*

Since (a) is classical and (e) derives from the preceding theorem, let me only sketch the proofs of (b), (c), (d). In any case Geza Freud observes that

$$\frac{f(x+h)-f(x)}{h} = -\sum_{k=1}^{l} n_k r_k \sin(n_k x + \varphi_k) + O(1) \tag{5}$$

if $h \approx n_l^{-1}$, that is $l \approx \log 1/|h|$.

For (b) we have to prove that

$$\overline{\lim_{l\to\infty}} \frac{1}{l} \sum_{k=1}^{l} n_k r_k \sin(n_k x + \varphi_k) > 0 \tag{6}$$

except on a set of the first category. Actually it is known that for some $C(q) > 0$ we have

$$\sup_{x\in I} \sum_{j=k}^{l} \rho_j \sin(n_j x + \varphi_j) > C(q) \sum_{j=k}^{l} \rho_j \tag{7}$$

whenever I is an interval of length larger than n_k^{-1} (Weiss [7], J.-P. Kahane, G. and M. Weiss [8]), and $\rho_j > 0$, therefore

$$\overline{\lim_{l\to\infty}} \left(\sum_{j=1}^{l} \rho_j \sin(n_j x + \varphi_j) \Big/ \sum_{j=1}^{l} \rho_j \right) > C(q)$$

or a dense G_δ-set whenever $\sum_{j=1} \rho_j = \infty$. Choosing $\rho_j = n_j r_j$ we have (6), that is (b).

For (c) what we have to prove is

$$0 < \overline{\lim_{l\to\infty}} \left((1/\sqrt{l \log\log l}) \sum_{k=1}^{l} n_k r_k \sin(n_k x + \varphi_k) \right) < \infty \tag{8}$$

for almost every x. This is Mary Weiss's law of the iterated logarithm [9].

For (d) we want to prove

$$\sum_{k=1}^{l} n_k r_k \sin(n_k x + \varphi_k) = O(1) \tag{9}$$

for a dense set of x. If q is large it is easy. In the general case we divise $\{n_k\}$ into a finite number of sequences with a large lacunarity constant. This is copied from Zygmund [10]. It finishes the proofs.

The theorem of Zygmund [10] deals with a Hadamard lacunary trigonometric series

$$\sum_{k=1}^{\infty} \rho_j \sin(n_j x + \varphi_j)$$

with $\rho_j \to 0$ (not necessarily a Fourier series). Such a series, the theorem says, converges on a dense set of x. Let us state an easy consequence.

THEOREM 3. *If $r_k = o(n_k^{-1})$, $f(x)$ is differentiable on a dense set.*

The proof comes from the fact that (5) holds with $o(1)$ instead of $O(1)$, due to the assumption $n_k r_k = o(1)$.

I have already mentioned the importance of Theorem 1. It is the easiest way to understand why the series

$$\sum_{k=0}^{\infty} b^{-k} \cos b^k x \tag{10}$$

(b integer $\geqslant 2$) represents a nowhere differentiable function.

Now let me explain the importance of Theorem 2; we may suppose that $f(x)$ is given by (10). I shall insist on (b), (c), and (d). From measure theoretical point of view, (c) is the ordinary behaviour. Considering x as the time, the ordinary points are defined by (c), the rapid points by (b), the slow points by (d). From (a) and (e) we see that the oscillation of f at x is as rapid as possible when x is a rapid point, as slow as possible when x is a slow point. The situation for (10) is not so simple as for

$$\sum_{k=0}^{\infty} b^{-k\alpha} \cos b^k x \tag{11}$$

when $0 < \alpha < 1$. In this case,

$$0 < \overline{\lim_{h \to 0}} \frac{|f(x+h) - f(x)|}{|h|^{\alpha}} < \infty$$

holds everywhere, and moreover $f \in \Lambda_\alpha$: all points are ordinary points.

It happens that ordinary points, rapid points, slow points occur for the Wiener function of Brownian motion (and for other Gaussian stationary processes as well). This was found by Orey and Taylor [11] for rapid points and by myself [12] for slow points, without being aware of the striking analogy with the case of series (10). For fine properties of slow points see Davis [13] and Perkins [14], for slow points of other processes Kahane [15]; see also [27].

There are many things in common between lacunary trigonometric series and random series. Here is an important comparison principle, due to Philipp and Stout [16]: assuming $A_k^2 = \frac{1}{2} \sum_{m=1}^{k} a_m^2 \to \infty$ and $a_k = o(A_k^{1-\delta})$ ($\delta > 0$), the random process defined by

$$S(t, \omega) = \sum_{k \leqslant N} a_k \cos(2\pi n_k \omega)$$

when $A_N^2 \leqslant t < A_{N+1}^2$, $\omega \in \Omega = [0, 1]$ has a version (on a larger probability space) which satisfy

$$S(t) - X(t) = o(t^{1/2-\lambda}),$$

where $X(t)$ is the Wiener function of Brownian motion, and $\lambda > 0$. This explains why the central limit theorem and the law of the iterated logarithm (used in the proof of Theorem 2) hold for partial sums of lacunary series. The lacunarity condition can be relaxed: see Takahashi [17]. If on the contrary the lacunarity condition is reinforced, more precise results are available: see Hawkes [18]. For similar results see Berkes [19], Kaufman and Philipp [20], Dudley and Hall [21], Kaufman [22], Murai [23]. In this last paper, Murai proves that the central limit theorem holds when $a_k = 1$ and $n_k = [\exp \sqrt{k}]$, a long standing question introduced by P. Erdös.

As an application of Hawkes results (Theorem 5 in Hawkes [18]) it can be checked that the set of x where (b) holds in Theorem 2 has Hausdorff dimension 1 when $f(x)$ is given by (10). This seems very likely in the general case. Generally speaking, a lot of results on $f(x+h) - f(x)$ can be derived from results on partial sums of lacunary trigonometric series and from Geza Freud's comparison principle (Theorem 1 in Freud [1]). Some old problems can be attacked anew. For example, Hardy 1916 was interested in distinguishing the cases when $|(f(x+h) - f(x))/h|$ diverges boundedly. This has very much to do with Hawkes 1980.

Though the relation with G. Freud is not so apparent let me mention two directions in which lacunary series were studied vigorously in the last few years.

The first and most important is the relation between lacunary trigonometric series and random trigonometric series. This is fundamental in the modern theory of Sidon sets, developed by S. Drury, D. Rider, G. Pisier, J. Bourgain, and many other (for references on Sidon sets see Myriam Déchamp [24]).

The second is the range of lacunary Taylor series inside the disc of convergence. The old Paley conjecture (the range is the whole plane except when the series converges absolutely on the boundary) was solved by Murai [25].

References

1. G. Freud, Über trigonometrische Approximation und Fouriersche Reihen, *Math. Z.* **78** (1962), 252–262.
2. G. Freud, On Fourier series with Hadamard gaps, *Studia Sci. Math. Hungar.* **1** (1966), 87–96.

3. M. Izumi, S.-I. Izumi, and J.-P. Kahane, Théorèmes élémentaires sur les séries de Fourier lacunaires, *J. Anal. Math.* **14** (1965), 235–246.
4. M. E. Noble, Coefficient properties of Fourier series with a gap condition, *Math. Ann.* **128** (1954), 55-62.
5. M. Sato, Lacunary Fourier series I, *Proc. Japan Acad.* **31** (1955), 402-405.
6. G. Benke, Lacunarity and Lipschitz properties on totlally disconnected abelian groups, *Michigan Math. J.* **25** (1978), 163–171.
7. M. Weiss, Concerning a theorem of Paley on lacunary power series, *Acta Math.* **102** (1959), 225–238.
8. J.-P. Kahane, M. Weiss, and G. Weiss, On lacunary power series, *Ark. Mat.* **5** (1963), 1–26.
9. M. Weiss, On the law of the iterated logarithm for lacunary trigonometric series, *Trans. Amer. Math. Soc.* **91** (1959), 444–469.
10. A. Zygmund, On the convergence of lacunary trigonometric series, *Fund. Math.* **16** (1930), 90–107.
11. S. Orey and S. J. Taylor, How often on a Brownian path does the law of iterated logarithm fail? *Proc. London Math. Soc.* **28** (3) (1974), 174–192.
12. J.-P. Kahane, Sur l'irrégularité locale du mouvement brownien, *C. R. Acad. Sci. Paris* **278** (1974), 331–333.
13. B. Davis, On Brownian slow points, *Z. Wahrsch. Verw. Gebiete* **64** (1983), 359–367.
14. E. Perkins, On the Hausdorff dimension of the Brownian slow points, *Z. Wahrsch. Verw. Gebiete* **64** (1983), 369–399.
15. J.-P. Kahane, Slow points of gaussian processes *in* "Conference on Harmonic Analysis in Honor of Antoni Zygmund," Vol. 1, The Wasworth Math. series, pp. 67–83, 1983.
16. W. Philipp and W. Stout, Almost sure invariance properties for partial sums of weakly dependent random variables, *Mem. Amer. Math. Soc.* **161** (1975).
17. S. Takahashi, Almost sure invariance principles for lacunary series, *Tohoku Math. J.* **31** (4) (1979), 439–451.
18. J. Hawkes, Probabilistic behaviour of some lacunary series, *Z. Wahrsch. Verw. Gebiete* **53** (1980), 21–33.
19. I. Berkes, On the central limit theorem for lacunary trigonometric series, *Anal. Math.* **4** (3) (1978), 159–180.
20. R. Kaufman and W. Philipp, A uniform law of the iterated logarithm for classes of functions *Ann. Probab* **6** (1978), 930–952.
21. E. Dudley and P. Hall, The Gaussian law and lacunary sets of characters, *J. Austral. Math. Soc. Ser. A* **27** (1) (1979), 91–107.
22. R. Kaufman, On the approximation of lacunary series by Brownian motion, *Acta Math. Acad. Sci. Hungar.* **35** (1980), 1–2; 61–66.
23. T. Murai, The central limit theorem for trigonometric series, *Nagoya Math. J.* **87** (1982), 79–94.
24. M. Dechamps, Sur les compacts associés aux ensembles lacunaires, les ensembles de Sidon et quelques problèmes ouverts. Groupe de travail sur les espaces de Banach invariants per translation, exp. no. 7, pp. 84–101, Publ. Math. Orsay 84.01, 1984.
25. T. Murai, Une conjecture de Paley sur les séries de Taylor lacunaires, *C. R. Acad. Sci. Paris* **290** (1980), 947–948.
26. J.-P. Kahane, Lacunary Taylor and Fourier Series, *Bull. Amer. Math. Soc.* **70** (1964), 199–213.
27. B. Davis and E. Perkins, Brownian slow points: the critical case, *Ann. Prob.* **13** (1985), 779–803.